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The Comparison between the Methods of Solution for Cubic Equations in Shushu Jiuzhang and Risālah fil-barāhin 'alā masā'il ala-Jabr wa'l-Muqābalah

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1. Introdution

Recently, the comparison between certain topics in ancient Chinese and Muslim mathematics has become increasingly popular. Some of the topics which were found in the mathematical treatises of Chinese and Muslim mathematicians have great similarities; for example, square root extraction, cube root extraction, one hundred fowls problems, rule of double false position and the Chinese Remainder Problem. This issue of great similarities has prompted many researchers to do further research in the comparison between Chinese and Muslim mathematics in selected topics. One of the researchers in current years who has been actively involved in this area is Chemla, K., who has done a research regarding the comparison between Chinese and Muslim root extraction methods. The finding shows great similarities between the ancient Chinese and Muslim root extraction methods, although the integration between the Chinese and Muslim discussions in that topic is yet to be established. The purpose of this paper is to make a comparison between the ancient Chinese and Muslim methods of solution for cubic equations; in particular, between *Shushu Jiuzhang* (Mathematical Treatise in Nine Sections) and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah.* They were among the earliest existing texts that discussed about the methods of solving cubic equations. The authors for both of the treatises were Qin Jiushao 秦九韶 and Umar Khayyam respectively, who were great mathematicians in their era.

2. The History of Cubic Equation in Ancient Chinese and Muslim Mathematics

The method of solution for cubic equations by the Chinese mathematicians was basically the extension of their method of cube root extraction which was first found in a first century treatise *Jiuzhang Suanshu* 九章算术 (Nine Chapters on the Mathematical Art), with a few modifications. In other words, the method of solution for cubic equations in Chinese mathematics was a special case of the solution of cube root extraction.

In China, the problems of cubic equations were first found in a treatise entitled *Jigu Suanjing* 缉古算经 (Continuation of Ancient Mathematics), by Wang Xiaotong

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王孝通 from the Tang Dynasty in the 7th century [1]. In *Jigu Suanjing*, Wang Xiaotong explained that the method of the solution of cubic equations was the method of cube root extraction. Thus in many ancient Chinese mathematical treatises, the method of the solution for cubic equations was not given, with the assumption that the readers were able to apply their mastery of the iteration method for cube root extraction to solve the cubic equations, as in Wang Xiaotong's case.

In China, the method of solution for cubic equations was found again in *Shushu Jiuzhang*, a 13th century mathematical treatise by Qin Jiushao. The method introduced in *Shushu Jiuzhang* was similar to the numerical solutions introduced by Paolo Ruffini in 1805 and W. G. Horner in 1819 [2].

Besides the solution for a cubic equation, the general method of solving higher degree equations was found for the first time and explained in detail in *Shushu Jiuzhang*. There were actually six equations of degree three or above in Shushu Jiuzhang: one of degree three, four of degree four and one of degree ten [3]. The only cubic equation in *Shushu Jiuzhang* was $4,608x^3 - 72,000,000,000,000 = 0$.

An ancient Muslim mathematical treatise that discussed in detail the method of solution for cubic equations is *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah* by 11th century mathematician Umar Khayyam. In his treatise, he explained that there were many Muslim mathematicians before and during his time who had pioneered the discussion of the method of solution for cubic equations. According to Umar Khayyam, they were Abu 'Abdullah Muhammad bin 'Isa al-Mahani (9th century), Abu Ja'far al-Khazin (10th century), al-Amir Abu Mansur Nasr bin 'Iraq (11th century), Abu'l-Jud bin Laith (11th century) and al-Hasan ibnu al-Haitham (11th century) [4].

However, the significance of the discussion of Umar Khayyam in *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*, which is still in existence today, is that he was the first Muslim mathematician to give a comprehensive explanation regarding the method of solution of cubic equations by conic sections.

3. The Comparison between the Methods of Solution for Cubic Equations in *Shushu Jiuzhang* and *Risālah fil-barāhin 'ala masā'il alā-Jabr wa'l-Muqābalah*

The methods of solution for cubic equations in ancient mathematical treatises were written in words, with strings of long and complicated explanations. They did not have the modern algebraic symbols that we are using today. In this paper, the methods of solution for cubic equations in *Shushu Jiuzhang* and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah* [5] are translated into modern algebraic symbols to enable the reader to understand the methods introduced by these ancient mathematicians.

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In general, the Chinese mathematicians did not classify the cubic equations. Thus, their cubic equations can be written in general form as:

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0$$

where a_0 is either positive or negative while a_1, a_2 and a_3 are the coefficients which are real numbers.

Qin Jiushao always fixed the constant term a_3 as negative whenever he solved a cubic equation. However, it was his method to standardize his solution for the cubic equations and this step was not compulsory. In fact, for Zhu Shijie, another famous Chinese mathematician from that golden era of Chinese mathematics, the constant term could be positive or negative.

Unlike Qin Jiushao, Umar Khayyam classified the cubic equations into 19 types, which consist of two, three and four terms, as follow:

(1) $c = x^3$
(2) $cx^2 = x^3$ (this is the same as $cx = x^2$)
(3) $cx = x^3$ (this is the same as $c = x^2$)
(4) $x^3 + ax^2 = bx$ (this is the same as $x^2 + ax = b$)
(5) $x^3 + bx = ax^2$ (this is the same as $x^2 + b = ax$)
(6) $bx + ax^2 = x^3$ (this is the same as $b + ax = x^2$)
(7) $x^3 + bx = c$
(8) $x^3 + c = bx$
(9) $c + bx = x^3$
(10) $x^3 + ax^2 = c$
(11) $x^3 + c = ax^2$
(12) $c + ax^2 = x^3$
(13) $x^3 + ax^2 + bx = c$
(14) $x^3 + ax^2 + c = bx$
(15) $x^3 + bx + c = ax^2$
(16) $x^3 = bx + ax^2 + c$
(17) $x^3 = ax^2 + bx + c$
(18) $x^3 + bx = ax^2 + c$
(19) $x^3 + c = bx + ax^2$

where a, b and c are positive coefficients.

According to Umar Khayyam, all the cubic equations above could only be solved by using conic sections, except the first cubic equation $c = x^3$, and the five cubic equations (2) to (6), which could be converted into quadratic equations. The equation $c = x^3$ was a cube root extraction problem and its root could be found by numerical method [6], while the five converted quadratic equations could be solved by geometrical method similar to that of Al-Khawarizmi in *Kitāb al-Jabr wa'l-Muqābalah.*

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In his discussion, Umar Khayyam always fixed the coefficient of x^3 as 1. The purpose of Umar Khayyam classifying the cubic equations into 19 types was to enable him to choose a suitable conic section to solve a specific cubic equation.

Next, we proceed with the comparison between the methods of solving cubic equations in *Shushu Jiuzhang* and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*, by presenting the methods in both texts respectively. The steps involved in Qin Jiushao's method for solving problems involving cubic equations are given below: For $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, where x is the root for the cubic equation, and a_0 is either positive or negative while a_1, a_2 and a_3 are the coefficients which are real numbers, let h be the largest digit value for the root of the cubic equation, thus $x = h + y$. Qin Jiushao used a unique Chinese counting device known as the counting rods in the calculation steps to solve his cubic equations. The counting rods were arranged in four lines. The steps that were involved on the counting board were:

Step 1:

Arrange a_0, a_1, a_2 and a_3 in 4 lines, with L_1, L_2, L_3 and L_4 denoting the first, second, third and fourth lines respectively:

$$\begin{array}{ll} L_4 & a_3 \\ L_3 & a_2 \\ L_2 & a_1 \\ L_1 & a_0 \end{array}$$

Step 2:

The coefficient in L_1, a_0, is multiplied by the approximation of the largest digit of the root, h. The product a_0h is added to the coefficient in L_2, a_1.

$$\begin{array}{ll} L_4 & a_3 \\ L_3 & a_2 \\ L_2 & a_0h + a_1 \\ L_1 & a_0 \end{array}$$

Step 3:

The step above is repeated for the coefficients in both L_3 and L_4. Eventually the sum $a_0h^3 + a_1h^2 + a_2h + a_3$ will be obtained in L_4. If the sum $a_0h^3 + a_1h^2 + a_2h + a_3$ is 0, then h will be the root for $f(x) = 0$. If not, continue with step 4.

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$$\begin{array}{ll} L_4 & a_0h^3 + a_1h^2 + a_2h + a_3 \\ L_3 & a_0h^2 + a_1h + a_2 \\ L_2 & a_0h + a_1 \\ L_1 & a_0 \end{array}$$

Step 4:

Repeat Step 2 and Step 3 until the sum obtained in L_3 is $3a_0h^2 + 2a_1h + a_2$.

$$\begin{array}{ll} L_4 & a_0h^3 + a_1h^2 + a_2h + a_3 \\ L_3 & 3a_0h^2 + 2a_1h + a_2 \\ L_2 & 2a_0h + a_1 \\ L_1 & a_0 \end{array}$$

Step 5:

Repeat Step 2 until the sum obtained in L_2 is $3a_0h + a_1$.

$$\begin{array}{lll} L_4 & a_0h^3 + a_1h^2 + a_2h + a_3 & (A_3) \\ L_3 & 3a_0h^2 + 2a_1h + a_2 & (A_2) \\ L_2 & 3a_0h + a_1 & (A_1) \\ L_1 & a_0 & (A_0) \end{array}$$

Let

$$\begin{aligned} A_0 &= a_0, \\ A_1 &= 3a_0h + a_1, \\ A_2 &= 3a_0h^2 + 2a_1h + a_2, \text{ and} \\ A_3 &= a_0h^3 + a_1h^2 + a_2h + a_3. \end{aligned}$$

Therefore, A_0, A_1, A_2 and A_3 are the coefficients for the cubic equation, which has been transformed into:

$$f(y) = A_0y^3 + A_1y^2 + A_2y + A_3 = 0.$$

The same procedure from Step 1 to 5 is repeated in order to find the approximation for the second digit value of the root for the cubic equation. Thus, the root for the cubic equation will now become $x = h + h' + y'$ or $y = h' + y'$. At one stage, if the sum $a_0(h')^3 + a_1(h')^2 + a_2(h') + a_3$ in L_4 is equal to 0, h' is the second digit value for the root of $f(x) = 0$. If not, the procedure will be carried on. In other words,

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the iteration will be repeated until the remainder in L_4 is zero, then the actual root of $f(x) = 0$ will be obtained.

As mentioned, in *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah* Umar Khayyam divided his cubic equations into 19 types so that different method of conic sections could be applied to solve each case of cubic equations individually. Therefore, in the method of conic sections used by Umar Khayyam, there was no general method for the solution of cubic equations. Out of the 19 types, 13 types cannot be solved by other methods except conic sections [7]. The method of conic sections for the 13 types proposed by Umar Khayyam is given below:

Cubic Equations	Solution with Conic Sections
1. $x^3 + ax = b$	Parabola and half circle
2. $x^3 + c = ax$	Hyperbola and parabola
3. $x^3 = ax + c$	Hyperbola and parabola
4. $x^3 + ax^2 = c$	Hyperbola and parabola
5. $x^3 + c = ax_2$	Hyperbola and parabola
6. $x^3 = ax^2 + c$	Hyperbola and parabola
7. $x^3 + ax^2 + bx = c$	Hyperbola and half circle
8. $x^3 + ax^2 + c = bx$	Hyperbola and hyperbola
9. $x^3 + bx + c = ax^2$	Circle and hyperbola
10. $x^3 = ax^2 + bx + c$	Hyperbola and hyperbola
11. $x^3 + ax^2 = bx + c$	Hyperbola and hyperbola
12. $x^3 + bx = ax^2 + c$	Circle and hyperbola
13. $x^3 + c = bx + ax^2$	Hyperbola dan hyperbola

An example of how Umar Khayyam solved a cubic equation using the method of conic sections will be given, by citing a specific example $x^3 + ax = b$ from *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah.* The steps of solution by Umar Khayyam for the cubic equation $x^3 + ax = b$, where a and b are positive, are as follow:

Let m be the side of a square such that $m^2 = a$ for a box with height h and volume b. Therefore, a is the area of one of the six surfaces of the box.

The volume of the box $= b$,

$$\begin{aligned} \therefore\ b &= a \times h \\ b &= m^2 \times h \end{aligned}$$

Construct a parabola with vertex B, parameter m (see definition below), axis BZ, and draw a line segment of length h perpendicular to BZ at B. Using h as diameter describe a semicircle, let the parabola and the semicircle intersect at D. DE and

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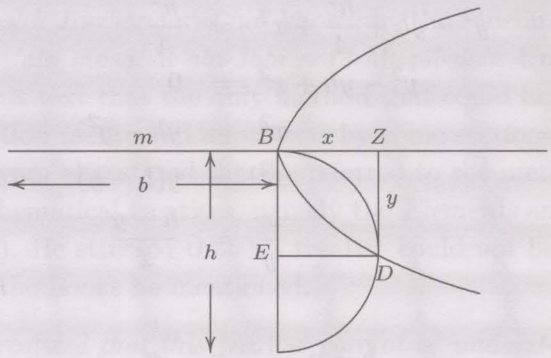
DZ, which are perpendicular to each other, are constructed to show the intersection point. Then $DZ = BE$ and with $y = BE$, it follows that $y^3 + ay = b$.



Umar Khayyam proved that the solution of each of the 13 cubic equations can be obtained from the intersection of two specific conic sections. In fact, he had always attempted to give proofs to all types of theorems that he was investigating. According to him:

> *I have always desired to investigate all types of theorems..., giving proofs for my distinctions, because I know how urgently this is needed in the solution of difficult problems.*

The proof that Umar Khayyam gave on the method of conic section involving the intersection of a parabola and circle as the solution for $x^3 + ax = b$ was as follow:
Let m be the parameter of the parabola, therefore $y^2 = mx$ (the definition of parabola by Apollonius, where m is a parameter) [9].
The equation of the circle is

$$\left(y - \frac{h}{2}\right) + x^2 = \left(\frac{h}{2}\right)^2.$$

Simplify the equation of the parabola:

$$\begin{aligned} y^2 &= mx \\ \frac{y}{m} &= \frac{x}{y} \end{aligned} \qquad (1)$$

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Simplify the equation of the circle:

$$\begin{aligned}
\left(y-\frac{h}{2}\right)^2+x^2 &= \left(\frac{h}{2}\right)^2\\
y^2-yh+\frac{h^2}{4}+x^2 &= \frac{h^2}{4}\\
y^2-yh+x^2 &= 0\\
x^2 &= yh-y^2\\
x^2 &= y(h-y)
\end{aligned}$$

(2) $$\frac{x}{y} = \frac{h-y}{x}$$

(1) = (2), therefore

$$\begin{aligned}
\frac{h-y}{x} &= \frac{x}{y}=\frac{y}{m}\\
\frac{m}{y} &= \frac{y}{x}=\frac{x}{h-y}\\
\frac{m^2}{y^2} &= \frac{m}{y}\times\frac{m}{y}\\
&= \frac{y}{x}\times\frac{x}{h-y}\\
&= \frac{y}{h-y}\\
m^2\times(h-y) &= y\times y^2\\
m^2h-m^2y &= y^3\\
y^3+m^2y &= m^2h
\end{aligned}$$

It is known that $a=m^2$ and $b=m^2h$ (from the assumption made in the beginning), therefore $y^3+m^2y=m^2h$ becomes $y^3+ay=b$, which is the form of the cubic equation that we are solving, with positive coefficients a and b.

Umar Khayyam proved that the intersection point of the parabola $y^2=mx$ and the semicircle $\left(y-\frac{h}{2}\right)^2+x^2=\left(\frac{h}{2}\right)^2$ was the solution for the cubic equation of the form $x^3+ax=b$, where a and b are positive coefficients with the assumption $a=m^2$ and $b=m^2h$. In *Shushu Jiuzhang* and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*, only positive solutions were accepted as the solutions for a cubic equation. In fact, all ancient Chinese and Muslim mathematicians only accepted positive solutions for their problems.

Umar Khayyam explained that for certain cases, there might be either more than one positive real root or no solution at all, depending on whether the conic sections that he was using intersected at more than one point or do not intersect. As in the

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case of $x^3 + ax = b$, the method of conic sections was also used by Umar Khayyam to find the solution for other types of cubic equations, as given in the earlier part of this discussion.

The method of iteration for the solution of cubic equation $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ in *Shushu Jiuzhang* was not known by the Muslim mathematicians. In *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*, Umar Khayyam stated in several parts of his text that the only method that could be used to solve the 13 types of cubic equation mentioned above was by conic sections.

In fact Umar Khayyam also stated that he referred to the method of conic sections in three Greek mathematical treatises, namely the *Elements* and *Data* (Euclid) and *Conics* (Apollonius). He stressed that his treatise could not be mastered if one did not refer to any of the books he mentioned.

> *It must be realized that this treatise cannot be understood except by one who has mastered Elements and his book called the Data and two books of Apollonius on the Conics, and anyone who does not know (any) one of these three cannot understand (the present treatise). And I have undertaken not to base myself in this treatise except on these three books* [10].

Therefore, it is not a surprising fact that there are great disparities in the methods of solution for cubic equation between *Shushu Jiuzhang* and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*. The reason is because the Chinese were never exposed to any Greek works in the 13th century, unlike the Muslim mathematicians, including Al-Khawarizmi who was well-known as the "Father of Aljabar", whose works were greatly influenced by the works of Greek mathematical treatises.

4. Conclusion

From the discussion in this paper, it can be concluded that there are great disparities in the methods of solving cubic equations between *Shushu Jiuzhang* and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah.*

Qin Jiushao and Umar methods of solving cubic equations were entirely different. Qin Jiushao used the iteration method while Umar Khayyam used the method of conic sections. Despite the time span of two centuries apart since the discussion of conic sections in *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*, the method of conic sections was still not known in China in the 13th century. In fact, the knowledge of ellipse, parabola and hyperbola was only known in China in the 17th century [11].

While Qin Jiushao provided a general method for solving the cubic and higher degree equations, Umar Khayyam applied the method of conic sections only to cubic equations and not to other higher degree equations because, according to him,

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the latter could not be solve by geometrical method. Besides that, Umar Khayyam had to divide the cubic equations into 19 types in order to apply a suitable method of conic section for each of the cubic equations. Therefore, Umar Khayyam did not give a general method or procedure for the solution of cubic equations.

Unlike Umar Khayyam who provided a proof for each of his case, Qin Jiushao did not provide any proof for his iteration method for solving his cubic and higher degree equations. While providing proof was a normal practice in ancient Muslim mathematical treatises, it was a rare practice in ancient Chinese mathematical treatises. In ancient Chinese mathematical treatises, a question was normally followed by the method of solution and last but not least, the answer.

In conclusion, it is clear that there is no integration at all in the discussion of the method for solving cubic equations between the Chinese and Muslim mathematics, in particular, between *Shushu Jiuzhang* and *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah*. This claim is strongly supported by the fact that *Risālah fi'l-barāhin 'alā masā' il ala-Jabr wa'l-Muqābalah* was greatly influenced by the works of Greek mathematical treatises. On the other hand *Shushu Jiuzhang* was never exposed to or influenced by any Greek mathematical treatises.

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